Oscillation of a Class of Fourth Order Dynamic Equations With Quasi Derivatives

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ABSTRACT

In this paper, the author studied, oscillatory and asymptotic behavior of bounded solutions of a class of fourth order dynamic equation with quasi-derivative of the form.

$$D_4\left(x(t)+f(t)x\big(\alpha_1(t)\big)\right)+g(t)G\left(x\big(\alpha_2(t)\big)\right)-h(t)H\left(x\big(\alpha_3(t)\big)\right)=0$$

and

$$D_4\left(x(t)+f(t)x\big(\alpha_1(t)\big)\right)+g(t)G\left(x\big(\alpha_2(t)\big)\right)-h(t)H\left(x\big(\alpha_3(t)\big)\right)=k(t)$$

for $t\in[t_0,\infty)_{\mathbb{T}}$, $\mathbb{T}$ is a time scale with $\sup\mathbb{T}=+\infty$, $t_0(\geq 0)\in\mathbb{T}$ are studied under the assumption

$$\int_{t_0}^{\infty}\frac{1}{p_n(t)}\Delta t<\infty, \qquad n=1,2,3$$

for various ranges of $f(t)$, where $D_n u(t)=p_n D_{n-1}^{\Delta}u(t), n=1,2,3$.

Keywords: *dynamic equation; neutral; delay; asymptotic; oscillation; positive and negative coefficients; bounded solution.*

1. Introduction

Kusano and Naito [2] studied a differential equation of the form

$$\big(r(t)y''(t)\big)''+y(t)F(y^2(t),t)=0$$

under the following assumptions.

(i) $r(t)$ is positive and continuous for $t\geq t_0$.

(ii) y(t) F(y[2](t),t) is continuous for $|y(t)|<\infty, t\geq 0$ and F(z, t) is positive for $z>0, t\geq t_0$ and $\int_{t_0}^{\infty}\frac{1}{r(t)}dt<\infty$.

In [3], Panigrahi and Ramireddy have studied the oscillatory and asymptotic behavior of solutions of

$$\left(r(t)\big(y(t)+p(t)y(\alpha(t))\big)^{\Delta^2}\right)^{\Delta^2}+q(t)G\big(y(\beta(t))\big)=0 \qquad (1)$$

and

$$\left(r(t)\big(y(t)+p(t)y(\alpha(t))\big)^{\Delta^2}\right)^{\Delta^2}+q(t)G\big(y(\beta(t))\big)=f(t) \qquad (2)$$

for $t\in[t_0,\infty)_{\mathbb{T}}$, and in [4], Panigrahi et. al. have been studied the oscillatory and asymptotic behaviour of solutions of

$$\left(r(t)\big(y(t)+p(t)y(\alpha(t))\big)^{\Delta^2}\right)^{\Delta^2}+q(t)G\big(y(\beta(t))\big)-h(t)H\big(y(\gamma(t))\big)=0 \qquad (3)$$

and

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$$\left(r(t)\left(y(t)+p(t)y(\alpha(t))\right)^{\Delta^2}\right)^{\Delta^2}+q(t)G\left(y(\beta(t))\right)-h(t)H\left(y(\gamma(t))\right)=f(t) \quad (4)$$

for $t\in[t_0,\infty)_{\mathbb{T}}$, under the assumption

$$\int_{t_0}^{\infty}\frac{\sigma(t)}{r(t)}\Delta t<\infty.$$

Throughout this paper, for the standard notations and terminology from the time scale calculus will be used [5,6]. In this paper, we consider a class of fourth order dynamic equations with quasi derivative of the form

$$D_4\left(x(t)+f(t)x(\alpha_1(t))\right)+g(t)G\left(x(\alpha_2(t))\right)-h(t)H\left(x(\alpha_3(t))\right)=0 \qquad (5)$$

and

$$D_4\left(x(t)+f(t)x(\alpha_1(t))\right)+g(t)G\left(x(\alpha_2(t))\right)-h(t)H\left(x(\alpha_3(t))\right)=k(t) \quad (6)$$

for various ranges of $f(t)$, with the condition

$$(A_0)\ \int_{to}^{\infty}\frac{1}{p_n(t)}\Delta t<\infty, \qquad n=1,2,3$$

where $p_n\in C\left([t_0,\infty)_{\mathbb{T}},(0,\infty)\right)$ and $f,k\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$; $g,h\in C_{rd}\left([t_0,\infty)_{\mathbb{T}},(0,\infty)\right)$; $G,H\in C(\mathbb{R},\mathbb{R})$ satisfy $uG(u)>0$, $uH(u)>0$ for $u\neq 0$; G is non decreasing, H is bounded; and $\alpha_1,\alpha_2,\alpha_3\in C_{rd}(\mathbb{T},\mathbb{T})$ are strictly increasing functions such that

$$\lim_{t\to\infty}\alpha_1(t)=\lim_{t\to\infty}\alpha_2(t)=\lim_{t\to\infty}\alpha_3(t)=\infty, \alpha_1(t)\le t, \alpha_2(t)\le t, \alpha_3(t)\le t.$$

We define the time scale interval $[t_0,\infty)_{\mathbb{T}}=[t_0,\infty)\cap\mathbb{T}$. For (5) and (6), we define the quasi derivative as follows. Let $y(t)=x(t)+f(t)x(\alpha_1(t))$, $D_o y(t)=y(t)$, $D_1y(t)=p_1(t)\,D_o{}^{\Delta}y(t)$, $D_2y(t)=p_2(t)\,D_1{}^{\Delta}y(t)$, $D_3y(t)=p_3(t)\,D_2{}^{\Delta}y(t)$, $D_4y(t)=D_3{}^{\Delta}y(t)$. Clearly, if $p_2(t)=r(t)$ and $p_1(t)=p_3(t)=1$, then the equations (5) and (6) reduces to (3) and (4) respectively.

Let $t_{-1}=\inf_{t\in[t_0,\infty)_{\mathbb{T}}}\{\alpha_1(t),\alpha_2(t),\alpha_3(t)\}$. By a solution of $(5)/(6)$, we mean a function $x\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that $x(t)+f(t)x(\alpha_1(t))$ is continuously delta differentiable, D_1, D_2, D_3 are differentiable operators and $(5)/(6)$ are satisfied for $t\ge t_0$. A solution of $(5)/(6)$ is said to be oscillatory if there exists a sequence $\{s_n\}$ in $[t_0,\infty)_{\mathbb{T}}$ such that $x(s_n)x(\sigma(s_n))\le 0$; Otherwise, it is called non oscillatory.

2. Existence Lemmas and Remarks.

This section deals with the existence Lemma, Remarks and the conditions which can be used to prove the theorems in section 3.

LEMMA 1. [3] Let $F,H,p:[t_0,\infty)_{\mathbb{T}}\to\mathbb{R}$ such that
$F(t)=H(t)+p(t)H(\alpha(t))$ for $t\in[t^{\wedge},\infty)_T$
where $\hat{t}\in[t_0,\infty)_{\mathbb{T}}$ and $\alpha(t)\ge t_0$ for all $t\in[\hat{t},\infty)_{\mathbb{T}}$. Assume that, there exists numbers $p_1,p_2,p_3,p_4\in\mathbb{R}$ such that p(t) is one of the following ranges.

(1) $-\infty<p_1\le p(t)\le 0$ (2) $0<p(t)\le p_2\le 1$ (3) $1<p_3\le p(t)\le p_4<\infty$

Suppose that $H(t)>0$ for large $t\in[t_0,\infty)_{\mathbb{T}}$, $\lim_{t\to\infty}\inf H(t)=0$ and that $\lim_{t\to\infty}F(t)=L\in\mathbb{R}$ exists. Then $L=0$.

The following assumptions have been used to prove the theorems in Section 3.

(A_1) $\int_{t_0}^{\infty}\frac{1}{p_1(s_1)}\int_{s_1}^{\infty}\frac{1}{p_2(s_2)}\int_{s_2}^{\infty}\frac{1}{p_3(s_3)}\int_{s_3}^{\infty}h(\theta)\ \Delta\theta\Delta s_3\Delta s_2\Delta s_1<\infty$;

(A_2) $\int_{t_0}^{\infty}\frac{1}{p_1(s_1)}\int_{t_0}^{s_1}\frac{1}{p_2(s_2)}\int_{t_0}^{s_2}\frac{1}{p_3(s_3)}\int_{t_0}^{s_3}g(\theta)\ \Delta\theta\Delta s_3\Delta s_2\Delta s_1=\infty$;

(A_3) $\int_{t_0}^{\infty}\frac{1}{p_2(s_2)}\int_{t_0}^{s_2}\frac{1}{p_3(s_3)}\int_{t_0}^{s_3}g(\theta)\ \Delta\theta\Delta s_3\Delta s_2=\infty$;

(A_4) $\int_{t_0}^{\infty}\frac{1}{p_3(s_3)}\int_{t_0}^{s_3}g(\theta)\ \Delta\theta\Delta s_3=\infty$;

(A_5) $G(-u) = -G(u), H(-u) = -H(u)$ for all $u \in R$;
(A_6) $\int_{t_0}^{\infty} g(\theta)\Delta\theta = \infty$.

REMARK 1.
Note that, (A_2) and (A_0) implies (A_3), (A_3) and (A_0) implies (A_4) and also (A_4) and (A_0) implies (A_6).
We will prove that, (A_2) and (A_0) implies (A_3) (other two results can be proved similarly, hence omitted). Assume, (A_2) and (A_0) hold, but (A_3) does not hold. That is
$\int_{t_0}^{\infty} \frac{1}{p_2(s_2)} \int_{t_0}^{s_2} \frac{1}{p_3(s_3)} \int_{t_0}^{s_3} g(\theta)\ \Delta\theta\Delta s_3\Delta s_2 < \infty.$
Let the limit be l_0. By the definition of limit, for each $\epsilon > 0$, there exists $s_1 > t_0$ such that
$\int_{t_0}^{s_1} \frac{1}{p_2(s_2)} \int_{t_0}^{s_2} \frac{1}{p_3(s_3)} \int_{t_0}^{s_3} g(\theta)\ \Delta\theta\Delta s_3\Delta s_2 < l_0 + \epsilon.$
Multiplying both sides by $\frac{1}{p_1(s_1)}$ and integrating the inequality from T to t, we obtain
$0 \le \int_{T}^{t} \frac{1}{p_1(s_1)} \int_{T}^{s_1} \frac{1}{p_2(s_2)} \int_{T}^{s_2} \frac{1}{p_3(s_3)} \int_{T}^{s_3} g(\theta)\ \Delta\theta\Delta s_3\Delta s_2\Delta s_1 < (l_0 + \epsilon) \int_{T}^{t} \frac{1}{p_1(s_1)} \Delta s_1$.
Taking limit as t→∞, and using (A_0), we obtain

$$0 \le \int_{t_0}^{\infty} \frac{1}{p_1(s_1)} \int_{t_0}^{s_1} \frac{1}{p_2(s_2)} \int_{t_0}^{s_2} \frac{1}{p_3(s_3)} \int_{t_0}^{s_3} g(\theta)\ \Delta\theta\Delta s_3\Delta s_2\Delta s_1 < \infty$$

which is a contradiction to (A_2).

REMARK 2. Let u be a continuously delta-differentiable function on $[t_0,\infty)_{\mathbb{T}}$ such that D_1u, D_2u, D_3u are continuously delta-differentiable functions and $D_4u \le 0 (\not\equiv 0)$ for large t, then any one of the following eight cases (a)-(h) holds, where
(a) $D_1u(t) > 0$, $D_2u(t) > 0$ and $D_3u(t) > 0$;
(b) $D_1u(t) > 0$, $D_2u(t) < 0$ and $D_3u(t) > 0$;
(c) $D_1u(t) < 0$, $D_2u(t) < 0$ and $D_3u(t) > 0$;
(d) $D_1u(t) < 0$, $D_2u(t) < 0$ and $D_3u(t) < 0$;
(e) $D_1u(t) < 0$, $D_2u(t) > 0$ and $D_3u(t) > 0$;
(f) $D_1u(t) < 0$, $D_2u(t) > 0$ and $D_3u(t) < 0$;
(g) $D_1u(t) > 0$, $D_2u(t) > 0$ and $D_3u(t) < 0$;
(h) $D_1u(t) > 0$, $D_2u(t) < 0$ and $D_3u(t) < 0$

3. Oscillation Criteria for Homogenous Equation with $\int_{t_0}^{\infty} \frac{1}{p_n(t)} \Delta t < \infty$, $n = 1, 2, 3.$

In this section, we find sufficient conditions for obtaining the oscillatory and asymptotic behaviour of bounded solution of (5) with the condition (A_0).

THEOREM 1: Let $0 \le f(t) \le f_1 < 1$ or $1 < f_2 \le f(t) \le f_3 < \infty$ holds. If $(A_0) - (A_2)$ and (A_5) holds, then every bounded solution of (5) either oscillates or converges to zero as t→∞.

Proof: Assume the contrary that, $x(t)$ is a non-oscillatory bounded solution of (5) such that $x(t) > 0$ for $t \ge t_0$. Then there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t)$, $x(\alpha_1(t))$, $x(\alpha_2(t))$, $x(\alpha_3(t))$ are all positive for $t \ge t_1$. Setting

$$y(t) = x(t) + f(t)x(\alpha_1(t)) \quad (7)$$

and $j(t) = \int_{t}^{\infty} \frac{1}{p_1(s_1)} \int_{s_1}^{\infty} \frac{1}{p_2(s_2)} \int_{s_2}^{\infty} \frac{1}{p_3(s_3)} \int_{s_3}^{\infty} h(\theta) H\left(x(\alpha_3(\theta))\right) \Delta\theta\Delta s_3\Delta s_2\Delta s_1$ (8)

Notice that condition (A_1), and the fact that H is a bounded function implies that $j(t)$ exists for all t. Now, if we let

$$z(t) = y(t) - j(t) = x(t) + f(t)x(\alpha_1(t)) - j(t) \quad (9)$$

Then, (5) changes to

$$D_4z(t) = -g(t)G\left(x(\alpha_2(t))\right) \le 0 (\not\equiv 0) \quad (10)$$

for $t \in [t_1,\infty)_{\mathbb{T}}$. Consequently, $z(t)$, $D_1 z(t)$, $D_2 z(t)$, and $D_3 z(t)$ are monotonic functions on $[t_1,\infty)_{\mathbb{T}}$. Then any one of the cases $(a) - (h)$ holds. Since $j^{\Delta}(t) < 0$, so $j(t)$ is monotonic and also $\lim_{t\to\infty} j(t) = 0$. Since $x(t), f(t)$ and $j(t)$ are bounded, it follow that $y(t)$ is bounded. Consequently, $z(t)$ is also bounded and $\lim_{t\to\infty} y(t)$and $\lim_{t\to\infty} z(t)$ both exists and finite.

Case I. First, suppose that $z(t) < 0$ for $t \geq t_2 > t_1$, then $-\infty < \lim_{t\to\infty} z(t) \leq 0$.
If $-\infty < \lim_{t\to\infty} z(t) < 0$, then $-\infty < \lim_{t\to\infty} y(t) < 0$, which is a contradiction as $x(t)$ and $f(t)$ are positive. If $\lim_{t\to\infty} z(t) = 0$, then from (9), we have $\lim_{t\to\infty} y(t) = 0$. Since $x(t) \leq y(t)$, then $\lim_{t\to\infty} x(t) = 0$.

Case II. Next, suppose that $z(t) > 0$ for $t \geq t_2 > t_1$. Then $0 < \lim_{t\to\infty} y(t) < \infty$ as $\lim_{t\to\infty} j(t) = 0$. Since $z(t) > 0$, by Lemma 1, any one of the cases (a)-(h) hold. Our claim in all the cases, $\lim_{t\to\infty} \inf x(t) = 0$. If not, let the limit be $l_1 > 0$. For some $\epsilon > 0$, there exists $t_3 \geq t_2$ such that $x(t) > l_1 - \epsilon > 0$ for $t \geq t_3 > t_2, t_3 \in [t_2,\infty)_{\mathbb{T}}$. So, (10) implies
$D_4 z(t) \leq -g(t)G(l_1 - \epsilon)$ (or) $g(t)G(l_1 - \epsilon) \leq -D_4 z(t)$

Suppose one of the cases (a), (b), (c) and (e) hold. Then integrating the last inequality, from t_3 to t, we obtain

$$G(l_1 - \epsilon) \int_{t_3}^{t} g(\theta)\Delta\theta \leq -D_3 w(t) + D_3 w(t_3) < D_3 w(t_3)$$

Since $\lim_{t\to\infty} D_3 z(t) > 0$, then by taking the limit as $t \to \infty$ in the last inequality, we obtain
$\int_{t_3}^{\infty} g(\theta)\Delta\theta < \infty$, a contraction to (A_6). From Lemma 1, we have $\lim_{t\to\infty} y(t) = 0$. Since $x(t) \leq y(t)$, so $\lim_{t\to\infty} x(t) = 0$.
Suppose case (d) holds. In this case also, we claim that $\lim_{t\to\infty} \inf x(t) = 0$. If it is not possible, let the limit be l_2>0. For some $\epsilon > 0$, there exists $t_3 > t_2$ such that $x(t) > l_2 - \epsilon > 0$ for $t \geq t_3$. Hence, (10) implies
$g(t)G(l_2 - \epsilon) \leq -D_4 z(t))$

Integrating the last inequality from t_3 to t, we obtain

$$-D_3 z(t) \geq -D_3 z(t_3) + G(l_2 - \epsilon) \int_{t_3}^{t} g(\theta)\Delta\theta$$

Again, integrating the last inequality from t_3 to t, we obtain

$$-D_2 z(t) \geq D_2 z(t_2) + G(l_2 - \epsilon) \int_{t_3}^{t} \frac{1}{p_3(s)} \int_{t_3}^{s} g(\theta)\Delta\theta\Delta s$$

Further, integrating the proceeding inequality from t_3 to t, we obtain

$$-D_1 z(t) \geq -D_1 z(t_3) + G(l_2 - \epsilon) \int_{t_3}^{t} \frac{1}{p_2(s)} \int_{t_3}^{s} \frac{1}{p_3(u)} \int_{t_3}^{u} g(\theta)\Delta\theta\, \Delta u \Delta s$$

$$\geq G(l_2 - \epsilon) \int_{t_3}^{t} \frac{1}{p_2(s)} \int_{t_3}^{s} \frac{1}{p_3(u)} \int_{t_3}^{u} g(\theta)\Delta\theta\, \Delta u \Delta s$$

Again, integrating the above inequality from t_3 to v, we obtain

$$z(t_3) \geq G(l_2 - \epsilon) \int_{t_3}^{v} \frac{1}{p_1(t)} \int_{t_3}^{t} \frac{1}{p_2(s)} \int_{t_3}^{s} \frac{1}{p_3(u)} \int_{t_3}^{u} g(\theta)\Delta\theta\, \Delta u \Delta s \Delta t$$

which is a contradiction to (A_2). Hence, by Lemma 1, $\lim_{t\to\infty} y(t) = 0$. Thus, $\lim_{t\to\infty} x(t) = 0$. Suppose cases (f) or (g) hold. In this also, first we claim $\lim_{t\to\infty} \inf x(t) = 0$. By proceeding as in the previous cases, we have
$D_4 z(t) \leq -g(t)G(l_3 - \epsilon)$

Or
$-D_4z(t) \geq G(l_3 - \epsilon)g(t)$

Integrating, the last inequality from t_3 to t, we obtain

$$-D_3z(t) \geq -D_3z(t_3) + G(l_3 - \epsilon) \int_{t_3}^{t} g(\theta)\Delta\theta$$

Again, integrating the preceeding inequality from t_3 to t, we get

$$D_2z(t_3) \geq D_2z(t) + G(l_3 - \epsilon) \int_{t_3}^{t} \frac{1}{p_3(s)} \int_{t_3}^{s} g(\theta)\Delta\theta\Delta s \geq G(l_3 - \epsilon) \int_{t_3}^{t} \frac{1}{p_3(s)} \int_{t_3}^{s} g(\theta)\Delta\theta\Delta s$$

We get a contradiction, due to (A_4). Thus, by Lemma 1, $\lim_{t\to\infty} y(t) = 0$. So, $\lim_{t\to\infty} x(t) = 0$. Suppose case (h) holds, in this case also, we claim $\lim_{t\to\infty} inf\ x(t) = 0$. By proceeding as in the previous cases, we have

$$-D_4z(t) \geq G(l_4 - \epsilon)g(t)$$

Integrating the last inequality from t_3 to t, we obtain

$$-D_3z(t) \geq -D_3z(t_3) + G(l_4 - \epsilon) \int_{t_3}^{s} g(\theta)\Delta\theta$$

Again, integrating the last inequality from t_3 to t, we get

$$-D_2z(t) \geq -D_2z(t_3) + G(l_4 - \epsilon) \int_{t_3}^{t} \frac{1}{p_3(\theta)} \int_{t_3}^{\theta} g(u)\Delta u\Delta\theta$$

Further, again integrating the proceeding inequality from t_3 to t, we obtain

$$D_1z(t_3) \geq D_1z(t) + G(l_4 - \epsilon) \int_{t_3}^{t} \frac{1}{p_2(s)} \int_{t_3}^{s} \frac{1}{p_3(\theta)} \int_{t_3}^{\theta} g(u)\Delta u\, \Delta\theta\Delta s$$

This implies,

$$D_1z(t_3) \geq G(l_4 - \epsilon) \int_{t_3}^{t} \frac{1}{p_2(s)} \int_{t_3}^{s} \frac{1}{p_3(\theta)} \int_{t_3}^{\theta} g(u)\Delta u\, \Delta\theta\Delta s$$

Since $\lim_{t\to\infty} D_1z(t)$ exists, we get a contradiction due to (A_3). Thus, by Lemma 1, $\lim_{t\to\infty} y(t) = 0$ and also $\lim_{t\to\infty} x(t) = 0$.
Finally, we suppose that $x(t) < 0$ for $t \geq t_0$. By putting,
w(t)= -x(t) for $t \geq t_0$ and by using (A_5), we obtain $w(t) > 0$ and

$$D_4\left(w(t) + p(t)w(\alpha_1(t))\right) + g(t)G\left(w(\alpha_2(t))\right) - h(t)H\left(w(\alpha_3(t))\right) = 0$$

Proceeding same as $x(t) > 0$ case, we obtain the desired results. Thus, completes the proof of the theorem.

EXAMPLE 1. Consider the fourth order differentiable equation

$$\left(e^t\left(e^t\left(e^t\left(x(t) + e^{-3}x(t-1)\right)'\right)'\right)'\right)' + e^{3t-45}x^5(t-3) -$$

$$e^{-9t-6}(1 + e^{-6t+12})\frac{x(t-2)}{1+x^2(t-2)} = 0 \qquad (11)$$

For t>t₂. Clearly, $(A_0) - (A_2)$ and (A_5) of Theorem 1 are satisfied. Hence, every bounded solution of (11) either oscillates or tends to zero as t→∞. Thus, $x(t) = e^{-3t}$ is such a bounded solution of (11), which converges to zero as t→∞.

<u>THEOREM 2</u>. Let $-\infty < f_4 \leq f(t) \leq f_5 < -1$ holds. If (A_0)-(A_2) and (A_5) hold, then every bounded solution of (5) either oscillates or converges to zero as t→∞.

Proof: Assume the contrary that, $x(t)$ is a bounded non oscillatory solution of (5) such that $x(t) > 0$ for $t \geq t_1$, $t_1 \geq t_0$. Using (7), (8) and (9), we obtain (10) for $t \geq t_1$. Consequently, $z(t), D_1z(t), D_2z(t), D_3z(t)$ are monotonic functions on $[t_1,\infty)_{\mathbb{T}}$. Then any one of the cases (a)-(h) hold. Since $j^{\Delta}(t) < 0$, so $j(t)$ is monotonic. Since $x(t), f(t)$ and $j(t)$ are bounded, it follows that $y(t)$ and $z(t)$ are bounded. Since $j(t)$ and $z(t)$ are monotonic, so $\lim_{t\to\infty} y(t)$ and $\lim_{t\to\infty} z(t)$ both exists and finite.

Case I. Suppose $z(t) > 0$ for $t \geq t_2$. Then $0 \leq \lim_{t\to\infty} z(t) < \infty$. If $0 < \lim_{t\to\infty} z(t) < \infty$, then proceeding same as in Case II of Theorem 1, we obtain $\lim_{t\to\infty} inf\ x(t) = 0$ and by Lemma 1, $\lim_{t\to\infty} y(t) = 0$. This implies that, $\lim_{t\to\infty} z(t) = 0$. Hence,

$$\begin{aligned} 0 = \lim_{t\to\infty} inf\ y(t) &\leq \lim_{t\to\infty} \inf(x(t) + f_5 x(\alpha_1(t))) \\ &\leq \lim_{t\to\infty} inf\big(x(t)\big) + f_5 \lim_{t\to\infty} sup\ x\big(\alpha_1(t)\big) \\ &= (1 + f_5)\lim_{t\to\infty} sup\ x(t) \qquad (12) \end{aligned}$$

Since $1 + f_5 < 0$, then $\lim_{t\to\infty} sup\ x(t) = 0$. Already, we have proved, $\lim_{t\to\infty} inf\ x(t) = 0$ and hence $\lim_{t\to\infty} x(t) = 0$.

Case II. Suppose, $z(t) < 0$ for $t \geq t_2 > t_1$. Then $-\infty < \lim_{t\to\infty} z(t) \leq 0$. If $-\infty < \lim_{t\to\infty} z(t) < 0$, then $-\infty < \lim_{t\to\infty} y(t) < 0$. Proceeding same as in Case II of Theorem 1, we can show that
$\lim_{t\to\infty} inf\ x(t) = 0$. Hence by Lemma 1, $\lim_{t\to\infty} y(t) = 0$. Thus, by (12), $\lim_{t\to\infty} x(t) = 0$.
Finally, we suppose that $x(t) < 0$ for $t \geq t_0$. By taking, w(t)= -x(t) for $t \geq t_0$ and by using (A_5), we obtain $w(t) > 0$ and

$$D_4\Big(w(t) + p(t)w\big(\alpha_1(t)\big)\Big) + g(t)G\Big(w\big(\alpha_2(t)\big)\Big) - h(t)H\Big(w\big(\alpha_3(t)\big)\Big) = 0$$

Proceeding same as $x(t) > 0$ case, we obtain the desired results. Thus, completes the proof of the theorem.

EXAMPLE 2. Consider the difference equation

$$\Delta\left(e^n\Delta\left(e^n\Delta\left(e^n\Delta\big(x(n) - e^3x(n-1)\big)\right)\right)\right) + e^{2n-9}x^5(n-1) - e^{-4n-6}\big(1 + e^{-6(n-2)}\big)\frac{x(n-2)}{1+x^2(n-2)} = 0, \quad n \geq 2 .$$

(13) Clearly, all the conditions of (A_0)-(A_2) and (A_5) of Theorem 2 are satisfied. Hence, every bounded solution of (13) is either oscillatory or converges to zero as t→∞. Thus, $x(n) = e^{-3n}$ is such a bounded solution of (13) , which converges to '0'.

<u>THEOREM 3</u>. Let $-1<f_6 \leq f(t) \leq 0$. If (A_0)-(A_2) and (A_5) hold, then every bounded solution of (5) either oscillates or converges to zero as t→∞.

Proof: Assume the contrary, that $x(t)$ is a bounded non oscillatory solution of (5) such that $x(t) > 0$ for $t \geq t_1 \geq t_0$. Using (7), (8) and (9), we obtain (10) for $t \geq t_1$. Consequently, $z(t), D_1z(t), D_2z(t), D_3z(t)$ are monotonic functions on $[t_1,\infty)_{\mathbb{T}}$. Thus
$z(t) > 0$ or < 0 for $t \geq t_2 \geq t_1$. Then any one of the cases (a)-(h) hold. Since $x(t), f(t)$ is bounded, it follows that $y(t)$ and $z(t)$ are bounded.

Case I. Suppose $z(t) > 0$ for $t \geq t_2$. Then $0 \leq \lim_{t\to\infty} z(t) < \infty$. If $0 < \lim_{t\to\infty} z(t) < \infty$, then proceeding same as in Case II of Theorem 1, we obtain $\lim_{t\to\infty} inf\ x(t) = 0$ and hence by Lemma 1, $\lim_{t\to\infty} y(t) = 0$. Hence,

$$\begin{aligned} 0 = \lim_{t\to\infty} sup\ y(t) &\geq \lim_{t\to\infty} sup\big(x(t)\big) + f_6 x\big(\alpha_1(t)\big) \\ &\geq \lim_{t\to\infty} sup\ x(t) + f_6 \lim_{t\to\infty} sup\ x\big(\alpha_1(t)\big) \\ &= \lim_{t\to\infty} sup\ x(t) + f_6 \lim_{t\to\infty} sup\ x\big(\alpha_1(t)\big) \end{aligned}$$

$$= (1+f_6)\lim_{t\to\infty} sup\, x(t) \qquad (14)$$

Since $1+f_6>0$, then $\lim_{t\to\infty} sup\, x(t)=0$. Already, we have prove that $\lim_{t\to\infty} inf\, x(t)=0$ and hence $\lim_{t\to\infty} x(t)=0$.

Case II. Suppose $z(t)<0$ for $t\geq t_2$. Then $-\infty<\lim_{t\to\infty} z(t)\leq 0$. If $-\infty<\lim_{t\to\infty} z(t)<0$, then $-\infty<\lim_{t\to\infty} z(t)\leq 0$. Proceeding same as in Case II of Theorem 1, we can show that $\liminf_{t\to\infty} x(t)=0$. Hence by Lemma I, $\lim_{t\to\infty} y(t)=0$. Thus by (14), we have $\lim_{t\to\infty} x(t)=0$. The case when $\lim_{t\to\infty} z(t)=0$, we have $\lim_{t\to\infty} y(t)=0$. So, $\lim_{t\to\infty} x(t)=0$ by (14).

EXAMPLE 3. Consider

$$\left(e^{t/3}\left(e^{t/3}\left(e^{t/3}\left(x(t)+\frac{e^{-1}}{2}x(t-1)\right)\right)'\right)'\right)'+\frac{e^{2t}}{e^{15}}x^5(t-3)-e^{-2-2t}(1+e^{-2t+4})\frac{x(t-2)}{1+x^2(t-2)}=0$$

(15)

for t ≥ 4. Clearly, all the conditions $(A_0)-(A_2)$ and (A_5) of Theorem 3 are satisfied. Hence, every bounded solution of (15) either oscillates or converges to zero as t→∞. Thus, $x(t)=e^{-t}$ is such a solution of (15), which converges to zero as t→∞.

4. Oscillation Criteria for Non-homogeneous Equations with $\int_{to}^{\infty}\frac{1}{p_n(t)}\Delta t<\infty$

This section is devoted to study the oscillatory and asymptotic behaviour of solutions of forced equation (6) with suitable forcing function.

(A_7) There exists a real valued continuously delta differentiable function F on$[t_0,\infty)_{\mathbb{T}}$ such that F(t) changes sign, p_1F^{Δ}, $p_2(p_1F^{\Delta})^{\Delta}, p_3(p_2(p_1F^{\Delta})^{\Delta})^{\Delta}$ are all real-valued continuously differentiable functions on $[t_0,\infty)_T$ and $(p_3(p_2(p_1F^{\Delta})^{\Delta})^{\Delta})^{\Delta}=k$ and $\lim_{t\to\infty} inf\, F(t)=-\infty, \lim_{t\to\infty} sup\, F(t)=\infty$.

THEOREM 4: Let $0\leq p(t)<p<\infty$ holds. Assume that (A_0), (A_1), (A_5) and (A_7) hold, then every bounded solution of (6) is oscillatory.

Proof: Let $x(t)$ be a non-oscillatory bounded solution of (6) such that $x(t)>0$ for $t\geq t_0$. Setting $y(t)$ as in (7), (8) and (9). Set

$$v(t)=z(t)-F(t)=y(t)-j(t)-F(t) \qquad (16)$$

For $t\geq t_1$, equation (6) becomes

$$L_4v(t)=-g(t)G\left(x\left(\alpha_2(t)\right)\right)\leq 0(\neq 0)$$

for $t\in[t_1,\infty)_{\mathbb{T}}$. Thus $v(t)$ is monotonic on $[t_1,\infty)_{\mathbb{T}}$.
Suppose $v(t)>0$ for $t\geq t_1$.Then 0 <$v(t)+j(t)=y(t)-F(t)$. Hence,

$$\lim_{t\to\infty} sup\, y(t)\geq \lim_{t\to\infty} sup\, F(t)\to\infty,$$

a contradiction to the fact that $x(t)$ is bounded and hence $y(t)$ is bounded.
Next, suppose $v(t)<0$ for $t\geq t_1$, then $z(t)<k(t)<F(t)$.
So, $\lim_{t\to\infty} inf\, y(t)=-\infty$, a contradiction to the fact $y(t)>0$. This completes the proof of the theorem.

EXAMPLE 4. Consider the fourth order differentiable equation

$$\left(e^{t}\left(e^{t}\left(e^{t}\left(x(t)-\frac{1}{2}x(t-6\pi)\right)'\right)'\right)'\right)'+(5e^{3t}+e^{-t}+e^{4t})x(t-2\pi)-e^{-t}(1+\cos^2 t)\frac{x(t-4\pi)}{1+x^2(t-4\pi)}=e^{4t}\cos t$$

(17)

for $t \geq 7\pi$. Clearly, all the conditions (A_0), (A_1), (A_5) and (A_7) of Theorem 4 are satisfied. Note that here $F(t) = -\frac{1}{170}e^t(\sin t + 4\cos t)$. Then, $(e^t(e^t(e^tF(t))')')' = k(t)$. Also, $\lim_{t\to\infty} sup\, F(t) = \infty$ and $\lim_{t\to\infty} inf\, F(t) = -\infty$. Hence, every bounded solution of (17) oscillates. Thus, $x(t) = \cos t$ is such a solution of (17), which oscillates.

5. Future work:

(i) For nonhomogeneous equations (6), we need study the oscillatory criteria for negative ranges of $f(t)$.

(ii) We need study the oscillatory criteria of solutions of a class of higher order dynamic equation with quasi-derivative of the form.

$$D_n\left(x(t) + f(t)x(\alpha_1(t))\right) + g(t)G\left(x(\alpha_2(t))\right) - h(t)H\left(x(\alpha_3(t))\right) = 0$$

and

$$D_n\left(x(t) + f(t)x(\alpha_1(t))\right) + g(t)G\left(x(\alpha_2(t))\right) - h(t)H\left(x(\alpha_3(t))\right) = k(t)$$

for $t \in [t_0,\infty)_{\mathbb{T}}$, where $\mathbb{T}$ is a time scale such that $\sup \mathbb{T} = +\infty$, $t_0(\geq 0) \in \mathbb{T}$ are studied under the assumption

$$\int_{t_0}^{\infty} \frac{1}{p_i(t)}\Delta t < \infty, \qquad i = 1, 2, 3, \dots, (n-1)$$

for various ranges of $f(t)$, where $D_i x(t) = p_i D_{i-1}^{\Delta} u(t),\ i = 1, 2, 3, \dots, n$.

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